How to Use Blocks and Polynomials to Multiply Three Digit (Base Six) Numbers

by Mark Karadimos, MATHguide.com [Updated December 15th, 2018]

Every wonder what would happen if humankind encountered an alien species? Imagine intelligent beings from a different planet (solar system, galaxy) whom are much different than us.

The need for our world's best linguists, cryptographers, physicists, and educators would be in high demand. We would need to understand their language to enable communication. Without being able to communicate mathematically would hinder our ability to communicate.

Enter mathematical operations with different bases.

It is likely that an alien civilization would not calculate their mathematics using base 10 mathematics. It is likely an alien civilization would not have members that have ten fingers like us. Maybe aliens could have developed their mathematics around base 2, like a computer, with only two characters to build numbers. Or, maybe they could have built their mathematics using six characters, which is called a base six system.

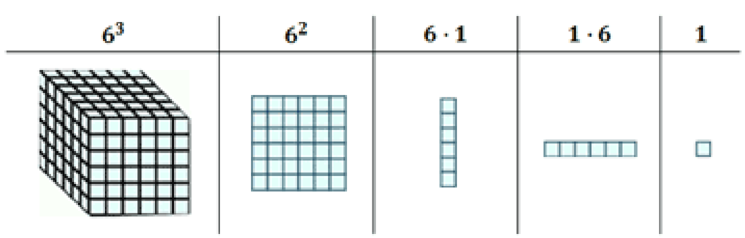
With that spirit in mind, this paper will explain how to multiply numbers using a base six system and it will demonstrate the process using base six blocks to help visualize the process.

What is **201 times 123**?

We have a method for doing this mathematics. Grammar school students use it. The algorithm involves splicing numbers into digits and multiplying them with the knowledge they exist within a certain place value.

Now, imagine changing the nature of the place value for the same problem but done in base six. These steps will explain how it is done.

First, here are the blocks we need to use and their representative value.



Reviewing the table above, we see how certain collections of squares represent values. 1 times 6 is viewed as an area, either 1 row times 6 columns or 6 rows times 1 column. 6-squared is viewed as 6 times 6 or a square of area 36 squares. However, 6-cubed is 6 times 6 times 6, which is literally a cube (volume) that has edges 6 units long.

Getting back to numbers, a table is required to organize them. Across the top of the table I will leave spaces for the three place values, base six. Remember, when we use our base ten system, our place values run like so for a three-digit number:

10^2	10^1	10^0

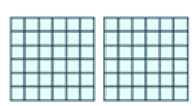
While we traditionally use the terms ones' place, tens' place, and hundreds' place for base ten, base six works the same way. Instead of a base ten, we swap the 10 for a 6. This leads us to think of place value according to this for base six:

6^2	6^1	6^0

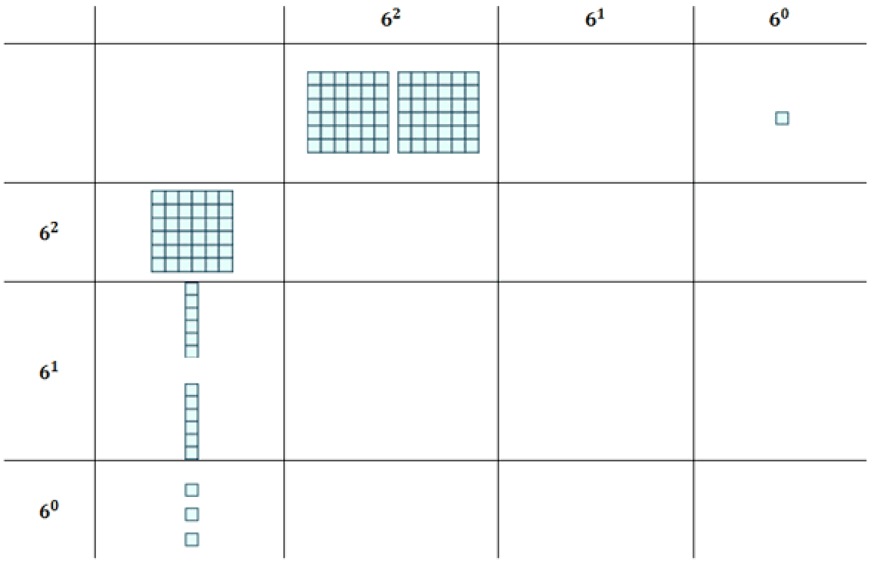
So, here is a table that can be used to multiply two three digit numbers, base 6.

		6^2	6^1	6^0
6^2				
6^1				
6^0				

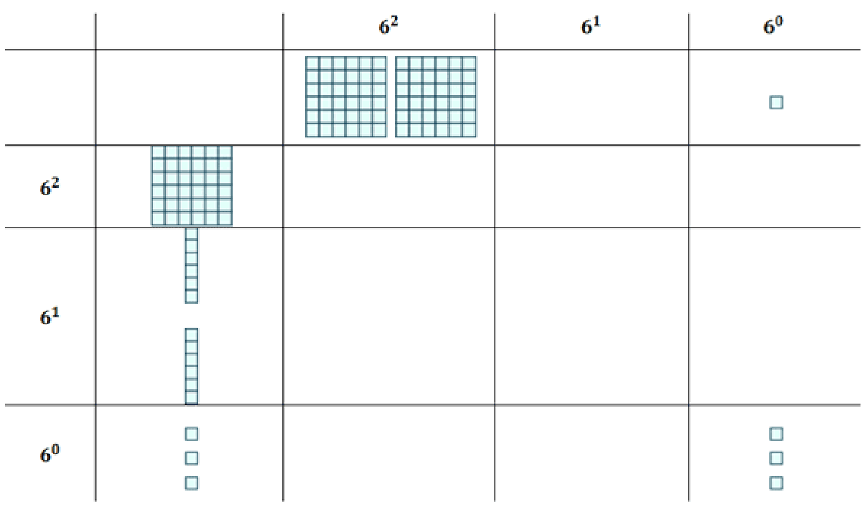
Now, the top row will be filled in with the first number, 201. Notice how 2 6x6 blocks represent the 2 in 201 and one block represents the 1 in 201 below.

		6^2	6^1	6^0
				
6^2				
6^1				

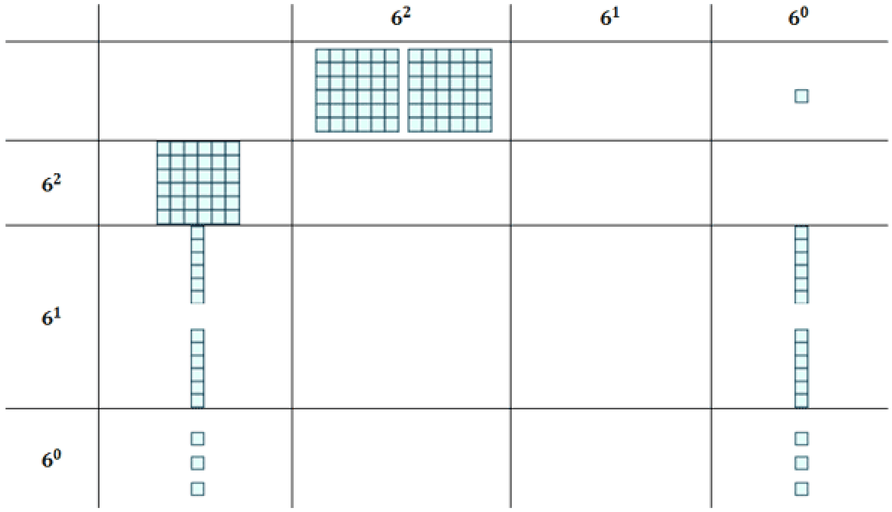
Now the left column will be filled in with the second number, 123. Notice the number of blocks is representative of the number according to their place value.



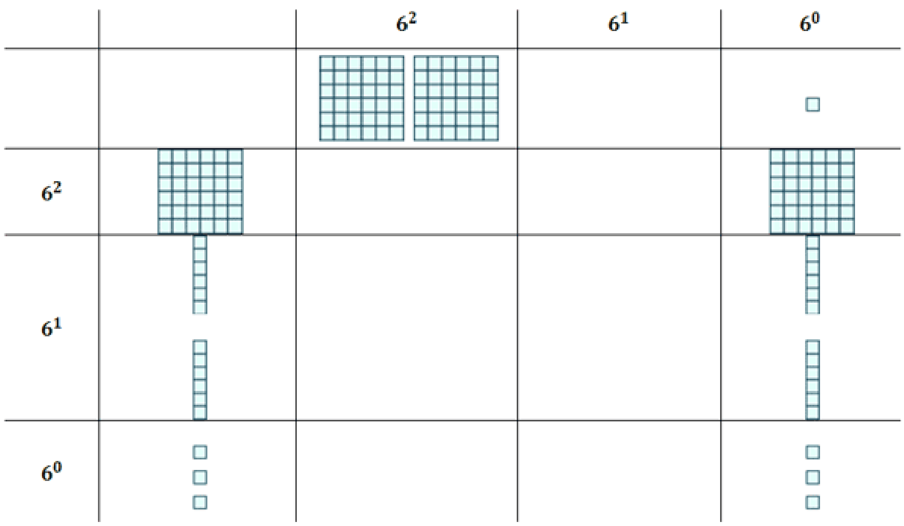
Next, multiplication begins. Let us multiply the numbers in the ones columns. 3 times 1 is 3. Look at the three blocks within the appropriate cell in the table below.



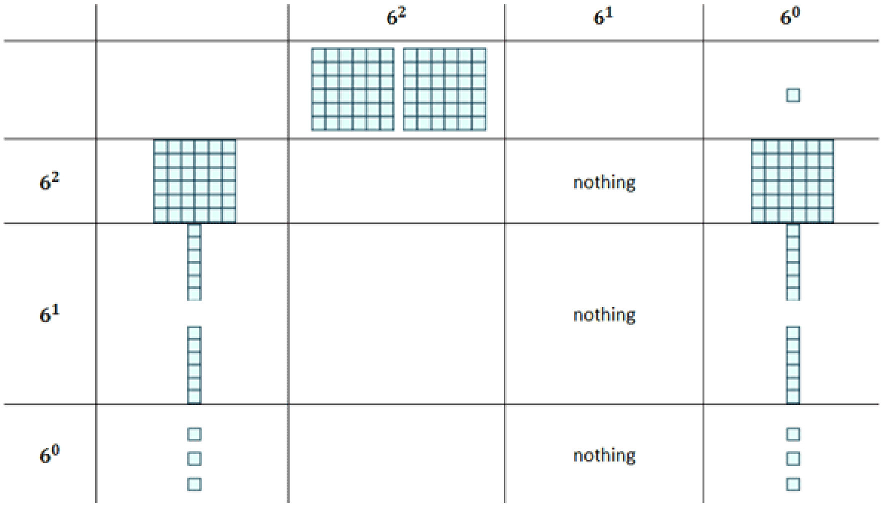
Next, multiply the left 6^1 digit times the top 6^0 digit. Like multiplication base 10, we use the simplified notion 2 times 1 is 2.



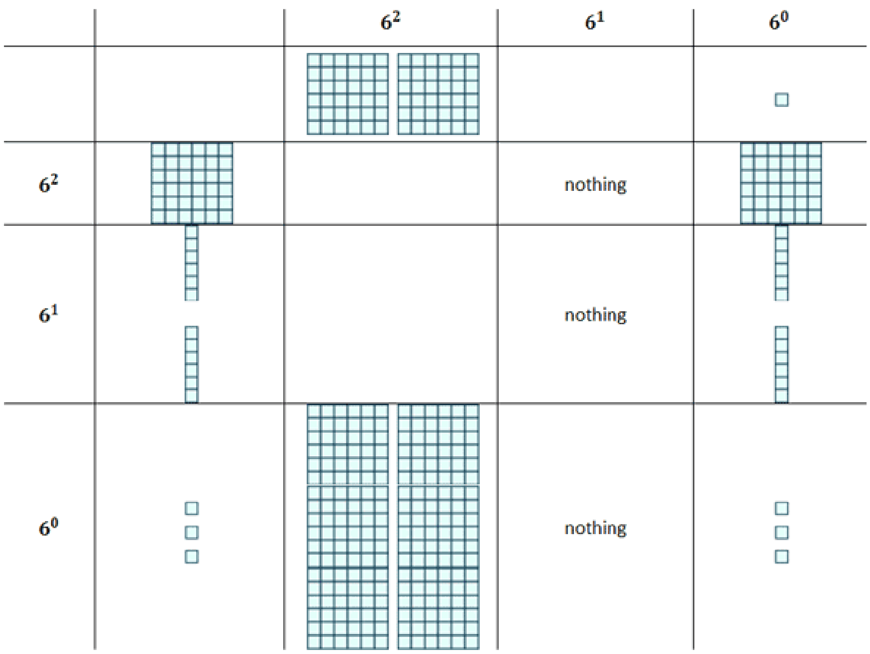
Next, multiply the left 6^2 digit times the top 6^0 digit.



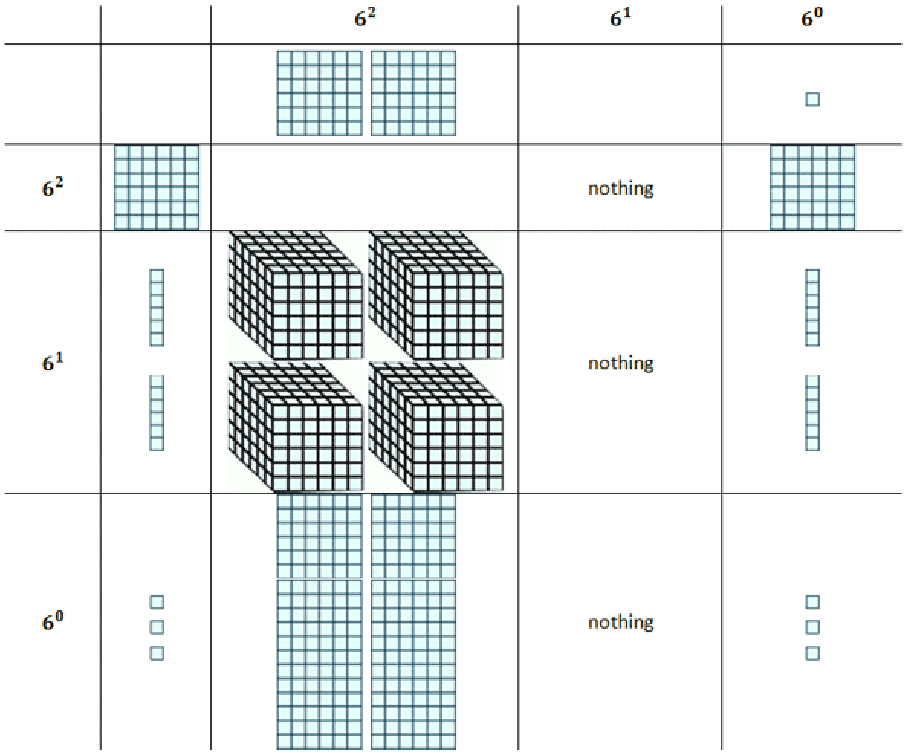
Next, multiply the left 6^0 digit times the top 6^1 digit to get nothing, seen below. The whole column is blank because we are multiplying by zero.



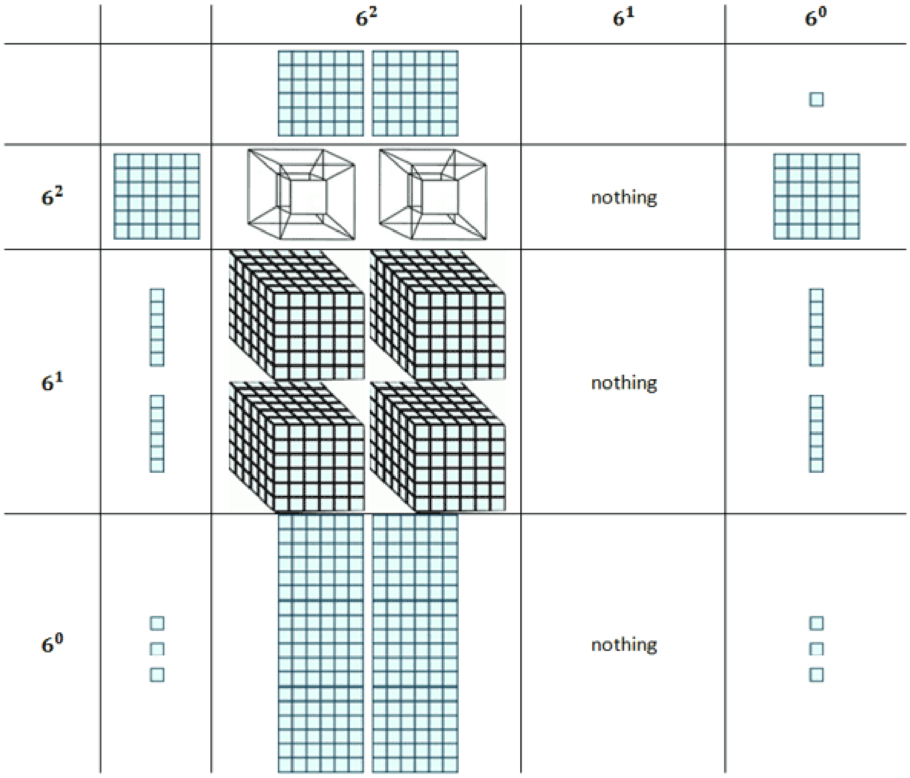
Next, multiply the left 6^0 digit times the top 6^2 digit to get...



Next, multiply the left 6^1 digit times the top 6^2 digit to get...



Next, multiply the left 6^2 digit times the top 6^2 digit to get 1 times 2 or 2 *hypercubes*. A hypercube is a four-dimensional cube. The images in the table are three dimensional shadows of those 4-dimensional objects. The dimensionality demands we move from cubes (3-dimensions) to the next higher dimension (4-dimensions), hypercubes.



The next steps relies on an ability to read the final diagram (last page). The diagram shows us:

Hypercubes	2
Cubes	4
Squares	7
Columns/Rows	2
Singles	3

Normally, this would give us the number 24723. However, there is no 7-digit in base 6. Base 6 mathematics relies only on six digits, 0 through 5. So, from the middlemost column (column with the 7), we must subtract 6 and consequently add one to the column to its left. This is routinely referred to as carrying, which is done in base 10 arithmetic.

The number 24723 becomes **25123**. This is the final answer. This value can be confirmed using an online base 6 calculator.

Let it be said the method used above reveals its limitation. As an area/volume representation, it is confusing because of the 4-dimensional nature of the problem. A hypercube would be a confusing concept. As a math education graduate student, it is confusing enough. Using this method to explain it to a grammar school student would be ill-advised.

This method can be seen to be mildly effective up to two-digit numbers. The system itself becomes too cumbersome to explain to lower ability students (or students in lower grade levels).

To make sense of this problem, it is more practical to revert to the generality of polynomials. The problem **201 times 123** can be adapted to $(2x^2+1)(x^2+2x+3)$. It requires thinking of the base as base-x instead of base 6. This has us doing the problem using the most general case.

Here is a table to organize the information.

	$\mathbf{2x^2}$		$\mathbf{1}$
$\mathbf{x^2}$			
$\mathbf{2x}$			
$\mathbf{3}$			

Next, we can fill it in, thinking of it as a multiplication table, like so.

	$\mathbf{2x^2}$		$\mathbf{1}$
$\mathbf{x^2}$	$2x^4$	0	x^2
$\mathbf{2x}$	$4x^3$	0	$2x$
$\mathbf{3}$	$6x^2$	0	3

The only part that requires some attention is the use of algebra within the table above. Recall this power property in mathematics:

$$x \cdot x = x^2$$
$$x \cdot x^2 = x^3$$
$$x^2 \cdot x = x^3$$
$$x^2 \cdot x^2 = x^4$$

Going back to the completed polynomial table above, we can collect all the terms within the table.

$$2x^4 + 4x^3 + 6x^2 + x + 2x + 3$$

We can combine like terms to get this simplified expression.

$$2x^4 + 4x^3 + 7x^2 + 2x + 3$$

Like numbers, we have to consider the nature of the base 6 system. There are only digits 0 - 5. So, 7 is really 7 - 6, and then we add 1 to the digit to its left. So, the x-squared term should get changed to x-squared and the x-cubed term should increase by one to 5 x-cubed.

$$2x^4 + 5x^3 + x^2 + 2x + 3$$

Therefore, the solution is: 25123 (base 6).